DIFFERENTIATION

Differentiation is the process of obtaining a derived function. The derived function is also known as the gradient function or derivative.

The gradient function or the derivative of a function has two steps.

(i) Multiply the function by the power.

(ii) The new function obtained after multiplication of the power its power is reduced by one.

Examples:

Differentiate the following:

(i) $y = x^5$

solution

step 1

$$\frac{dy}{dx} = 5x^5$$

step 2

$$\frac{dy}{dx} = 5x^{5-1}$$

$$= 5x^4$$

(ii) $y = x^6$

solution

step 1

$$\frac{dy}{dx} = 6x^6$$

step 2

$$\frac{dy}{dx} = 6x^{6-1}$$

$$= 6x^5$$

(iii) $y = x^9$

solution

step 1

$$\frac{dy}{dx} = 9x^9$$

step 2

$$\frac{dy}{dx} = 9x^{9-1}$$

$$= 9x^8$$

(iv) $y = x^{20}$

solution

step 1

$$\frac{dy}{dx} = 20x^{20}$$

step 2

$$\frac{dy}{dx} = 20x^{20-1}$$

$$= 20x^{19}$$

(v) $y = x^{-3}$

solution

step 1

$$\frac{dy}{dx} = -3x^{-3}$$

step 2

$$\frac{dy}{dx} = -3x^{-3-1}$$

$$= -3x^{-4}$$

(vi) $y = x^{1/2}$

solution

step 1

$$\frac{dy}{dx} = \frac{1}{2}x^{1/2}$$

step 2

$$\frac{dy}{dx} = \frac{1}{2}x^{1/2-1}$$

$$= \frac{1}{2}x^{-1/2}$$

(vii) $y = x^{-1/2}$

solution

step 1

$$\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$

step 2

$$\frac{dy}{dx} = -\frac{1}{2}x^{-1/2-1}$$

$$= -\frac{1}{2}x^{-1 1/2}$$

(viii) $y = 8$

The derivative of a constant will always be equal to zero. A constant is a number that does not have a variable.

$$y = 8$$

$$\frac{dy}{dx} = 0$$

Task

Differentiate the following

(i) $y = 8x^5$

(ii) $y = -4x^6$

(iii) $y = \frac{1}{2}x^4$

(iv) $y = -\frac{1}{8}x^{12}$

(v) $y = -\frac{1}{5}x^{10}$

(vi) $y = -\frac{1}{5}x^{-1/3}$

<u>solution</u>

Step 1

$$\frac{dy}{dx} = -\frac{1}{3} \times -\frac{1}{5}x^{-1/3}$$

$$= \frac{1}{15}x^{-1/3}$$

Step 2

$$\frac{dy}{dx} = \frac{1}{15}x^{-1/3 - 1}$$

$$= \frac{1}{15}x^{-1\frac{1}{3}}$$

(v) $y = 10x$

The derivative of a polynomial

A polynomial in x is an expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_n$ where n is a positive integer and $a_0, a_1, \ldots a_n$ are constants. For example $2x^3 + 4x^2 - 5x - 7$ is a polynomial.

Find the derivative of the following:

4 (i) $y = 4x^2 + 3x + 2$

solution

$$y = 4x^2 + 3x + 2$$

$$\frac{dy}{dx} = 2\times4x^{2-1} + 1\times3x^{1-1} + 0$$

$$= 8x + 3x^0$$

$$= 8x + 3.$$

(ii) $y = x^3 - 3x^2$

solution

$$y = x^3 - 3x^2$$

$$\frac{dy}{dx} = 3\times x^{3-1} - 2\times3x^{2-1}$$

$$= 3x^2 - 6x$$

(iii) $y = (x+2)(x-1)$

solution

$$y = (x+2)(x-1)$$

Open the brackets first

$$y = x(x-1) + 2(x-1)$$

$$y = x^2 - x + 2x - 2$$

$$y = x^2 + x - 2$$

$$\frac{dy}{dx} = 2x^{2-1} + 1\times x^{1-1} - 0 = 2x + x^0$$

$$= 2x + 1.$$

(iv) $y = \dfrac{x^3 + x^2 + x}{x}$

Divide each value in the denominator by x

$y = \dfrac{x^3}{x} + \dfrac{x^2}{x} + \dfrac{x}{x}$

$y = x^2 + x + 1$.

$\dfrac{dy}{dx} = 2x^{2-1} + 1 \times x^{1-1} + 0$

$= 2x + x^0$

$= 2x + 1$.

2 (a) Find the derived function of each of the following

(a) $s = 2t^3 - 3t^2 + 4t$

solution

$\dfrac{ds}{dt} = 3 \times 2t^{3-1} - 2 \times 3t^{2-1} + 1 \times 4t^{1-1}$

$= 6t^2 - 6t + 4t^0$

$= 6t^2 - 6t + 4$.

(b) $V = \frac{4}{3}\pi r^3$

solution

$V = \frac{4}{3}\pi r^3$

$\dfrac{dV}{dr} = 3 \times \frac{4}{3}\pi r^{3-1}$

$= 4\pi r^2$.

(c) $A = v^2 - 2v + 10$

solution

$A = v^2 - 2v + 10$

$\dfrac{dA}{dv} = 2v^{2-1} - 1 \times 2v^{1-1} + 0$

$= 2v - 2v^0 = 2v - 2$.

3 Differentiate the following with respect to x

(I) $y = x^4 - 3x^2 + 7$

solution

$y = x^4 - 3x^2 + 7$

$\frac{dy}{dx} = 4 \times x^{4-1} - 2 \times 3x^{2-1} + 0$

$= 4x^3 - 6x$

(II) $y = 5x^4 - 2x^5 - 7x^3$

solution

$y = 5x^4 - 2x^5 - 7x^3$

$\frac{dy}{dx} = 4 \times 5x^{4-1} - 5 \times 2x^{5-1} - 3 \times 7x^{3-1}$

$= 20x^3 - 10x^4 - 21x^2$

(III) $y = -\frac{1}{2}x^{20} - \frac{1}{9}x^9 + 20$

solution

$y = -\frac{1}{2}x^{20} - \frac{1}{9}x^9 + 20$

$\frac{dy}{dx} = 20 \times -\frac{1}{2}x^{20-1} - 9 \times \frac{1}{9}x^{9-1} + 0$

$= -10x^{19} - x^8$

(IV) $y = 7x^2 - \frac{1}{2}x$

solution

$y = 7x^2 - \frac{1}{2}x$

$\frac{dy}{dx} = 2 \times 7x^{2-1} - 1 \times \frac{1}{2}x^{1-1}$

$= 14x - \frac{1}{2}x^0 = 14x - \frac{1}{2}$

4 Find y' if

(a) $y = \frac{1}{10}x^5 + 7x^3 - 2x^2$

solution

$$y' = \frac{dy}{dx} = 5 \times \frac{1}{10} x^{5-1} + 3 \times 7x^{3-1} - 2 \times 2x^{2-1}$$

$$= \frac{1}{2} x^4 - 21x^2 - 4x$$

(b) $y = 4t^4 - \frac{1}{6}t^2 + 7t + 10$

solution

$$y' = \frac{dy}{dt} = 4 \times 4t^{4-1} - 2 \times \frac{1}{6} t^{2-1} + 1 \times 7t^{1-1} + 0$$

$$= 16t^3 - \frac{1}{3}t + 7t^0$$

$$= 16t^3 - \frac{1}{3}t + 7.$$

(c) $y = \frac{1}{24} r^{24} - \frac{1}{40} r^{10} - 9.$

solution

$$y' = \frac{dy}{dr} = 24 \times \frac{1}{24} r^{24-1} - 10 \times \frac{1}{40} r^{10-1} - 0$$

$$= r^{23} - \frac{1}{4} r^9.$$

5 Find the gradient of the given curves at the indicated points.

(a) $y = x^2 - 3x \quad (2, -2).$

solution:

$$\text{gradient} = \frac{dy}{dx} = 2 \times x^{2-1} - 1 \times 3x^{1-1}$$

$$= 2x - 3x^0 = 2x - 3$$

gradient $= 2x - 3$

The value of $x = 2$

gradient $= 2 \times 2 - 3$

$= 4 - 3 = 1$.

(b) $y = x^4 - 3x^2 + 6x \quad (1,4)$.

solution

gradient $= y' = \frac{dy}{dx}$

$= 4 \times x^{4-1} - 2 \times 3x^{2-1} + 1 \times 6x^{1-1}$

$= 4x^3 - 6x + 6x^0$

$= 4x^3 - 6x + 6$.

The value of $x = 1$

gradient $= 4x^3 - 6x + 6 = 4(1)^3 - 6(1) + 6$

$= 4 - 6 + 6$

$= 4$.

(c) $y = x^5 - 7x + 3 \quad (0,3)$

solution

$y = x^5 - 7x + 3$

$\frac{dy}{dx} = y' =$ gradient

$= 5x^{5-1} - 1 \times 7x^{1-1} + 0$

$= 5x^4 - 7x^0 = 5x^4 - 7$.

The value of $x = 0$.

gradient $= 5(0)^4 - 7$

$= -7$.

6 At what point is the gradient of

(a) $y = x^2 + 3x + 2$ equal to 9.

solution

$$\text{gradient} = \frac{dy}{dx} = 2 \times x^{2-1} + 1 \times 3x^{1-1} + 0$$

$$= 2x + 3x^0$$

$$= 2x + 3$$

gradient $= 9 \Rightarrow 2x + 3 = 9$

$$2x = 9 - 3$$

$$2x = 6$$

$$x = 3.$$

$y = x^2 + 3x + 2$ but $x = 3$

$$y = 3^2 + 3 \times 3 + 2 = 20$$

The point is $(3, 20)$

(b) $y = \frac{2}{3}x^3 + x^2$ gradient $= 4$

solution

$$\frac{dy}{dx} = \text{gradient} = 3 \times \frac{2}{3}x^{3-1} + 2 \times x^{2-1}.$$

$$= 2x^2 + 2x$$

gradient $= 4$

$$2x^2 + 2x = 4$$

$$2x^2 + 2x - 4 = 0$$

$$2x^2 - 2x + 4x - 4 = 0$$

$$2x(x - 1) + 4(x - 1) = 0$$

$$(2x + 4)(x - 1) = 0$$

$(x-1)=0$ or $2x+4=0$

$x=1$ or $2x=-4$

$x=-2$.

$y=\frac{2}{3}x^3+x^2$

when $x=1$

$y=\frac{2}{3}(1)^3+(1)^2$

$y=\frac{2}{3}+1$

$=1\frac{2}{3}$

The point is $(1, 1\frac{2}{3})$

When $x=-2$

$y=\frac{2}{3}(-2)^3+(-2)^2$

$y=-\frac{16}{3}+4$

$y=-1\frac{1}{3}$

The point is $(-2, -1\frac{1}{3})$

(c) $y=\frac{1}{4}x^4-2x^3+\frac{5}{2}x^2$ equal to 0.

<u>solution</u>

gradient $=\frac{dy}{dx}=4\times\frac{1}{4}\times x^{4-1}-3\times 2x^{3-1}+2\times\frac{5}{2}x^{2-1}$

$=x^3-6x^2+5x$

gradient $=0$

$x^3-6x^2+5x=0$.

$x(x^2-6x+5)=0$.

$x=0$ or $x^2-6x+5=0$.

$x^2-x-5x+5=0$.

$x(x-1)-5(x-1)=0$

$(x-5)(x-1) = 0$

$x-5=0$ or $x-1=0$

$x=5$ or $x=1$

When $x=0$, $y = \frac{1}{4}(0)^4 - 2(0)^3 + \frac{5}{2}(0)^2$

$= 0$

The point is $(0,0)$

When $x = 1$

$y = \frac{1}{4}(1)^4 - 2(1)^3 + \frac{5}{2}(1)^2$

$= \frac{1}{4} - 2 + \frac{5}{2}$

$= \frac{3}{4}$

The Point is $(1, \frac{3}{4})$

When $x = 5$

$y = \frac{1}{4}(5)^4 - 2(5)^3 + \frac{5}{2}(5)^2$

$= \frac{1}{4} \times 625 - 2 \times 125 + \frac{5}{2} \times 25$

$= \frac{625}{4} - 250 + \frac{125}{2}$

$= 218\frac{3}{4} - 250$

$= -31\frac{1}{4}$

The point is $(5, -31\frac{1}{4})$

7 (i) Find $\frac{dy}{dx}$ given that

$$y = \frac{x^3 - 3x^2}{x-3}$$

solution

$$y = \frac{x^3 - 3x^2}{x-3}$$

$$\cancel{\frac{dy}{dx}}\ y = \frac{x^2 (x-3)}{x-3}$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x^{2-1}$$

$$= 2x$$

(ii) $$y = \frac{x^2 + 4x + 4}{x+2}$$

solution

$$y = \frac{x^2 + 2x + 2x + 4}{x+2}$$

$$y = \frac{x(x+2) + 2(x+2)}{(x+2)}$$

$$y = \frac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}}$$

$$y = x+2$$

$$\frac{dy}{dx} = 1 \times x^{1-1} + 0$$

$$= x^0$$

$$= 1$$